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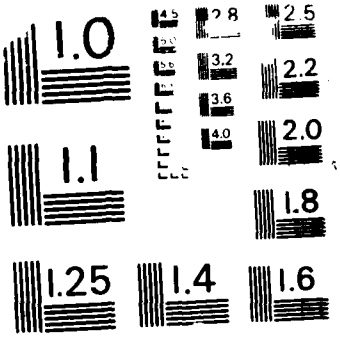

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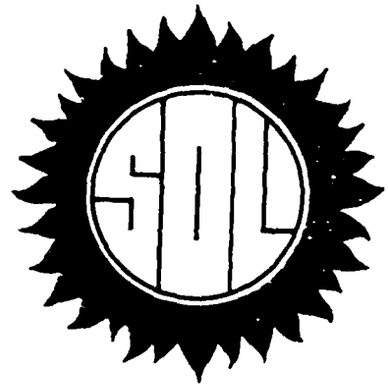

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Eliminating Columns in the Simplex Method for Linear Programming

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November 1987

Abstract

We propose a column-eliminating and a lower bound updating technique for the simplex method for linear programming. A pricing criterion is developed for checking whether or not a dual hyperplane corresponding to a column intersects a simplex containing all of the optimal dual feasible solutions. If the dual hyperplane has no intersection with this simplex, we can eliminate the corresponding column from the constraints. As the simplex method iterates, the working constraint matrix eventually eliminates all columns except those that are in at least one optimal basis.

Key words: Linear Programming, Simplex Method, Ellipsoid Method; Karmarkar Method, Column Elimination, Lower Bound Updating.

* The author is very grateful for many discussions with Professor George Dantzig and his invaluable suggestions about this work.

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1. Introduction

Techniques for solving linear programming (LP) have been intensely studied for four decades. The birth of linear programming is usually identified with the concurrent development of the simplex method in 1947 by George Dantzig. It says much for the algorithm's originator that the simplex method still remains the major algorithm used in optimization systems, although recently interior methods are serious competitors.

Two more recent approaches to solving linear programs are the ellipsoid method (Khachiyan [7]) and the projective algorithm (Karmarkar [6]). Recently, Todd [8] and I [9] found that the ellipsoid method and Karmarkar's algorithm are closely related. Both of these methods generate a shrinking dual ellipsoid containing the optimal dual solutions. Therefore, some columns may be eliminated from the constraints if their corresponding dual hyperplanes have no intersection to the containing ellipsoid.

When I talked with George Dantzig regarding this eliminating issue, he predicted that a similar result could be obtained for the simplex method. Therefore, we started a series of research meetings along this direction. The result is this paper on a criterion for eliminating columns in each iteration of the simplex method. In this report, we introduce the notion of a simplex S^t that contains all optimal feasible dual solutions. When the primal is in canonical form and feasible, the simplex in the dual space is the negative orthant and a half space whose boundary hyperplane is parallel to the dual objective. At the next iteration $t+1$, a separating dual hyperplane deletes part of the simplex S^t and forms a new containing simplex, S^{t+1}. Thus, we developed a strong column-eliminating theorem: if a dual hyperplane corresponding to column j has no intersection with the containing simplex S^t, then column j in the primal constraint matrix cannot be in an optimal basis and, therefore can be eliminated from further computation. As the simplex method iterates, the working constraint matrix ultimately reduces to only those columns that are in at least one optimal basis.

As Goldfarb and Hao [5] pointed out, the theorem derived in this paper is similar to the one proposed by Cheng [1, 2]. In particular, Corollary 1 in our paper is identical to Theorem 1 on identifying the permanent nonbasic variables for the simplex method in

Cheng [1]. A complete literature on identifying permanent nonbasic and basic variables can be found in Cheng [2]. However, all existing techniques need to approximate the optimal objective value, and we develop the theoretical basis for a lower bound updating technique. Most existing techniques need the assumption of nondegeneracy, and we relax this assumption after initially assuming it. Moreover, the geometrical interpretation of the containing simplex in our theorem is new and this interpretation may lead to a pricing rule for selecting the incoming column as the one that most reduces the volume of the containing simplex of iteration $t+1$ relative to that of iteration t. We also give a simple numerical example to show how this column-eliminating scheme performs in practice.

2. The Dual Containing Simplex

We assume for the moment:

Assumptions

1. an initial basic feasible solution is known;
2. the minimal objective value is known in advance;
3. every vertex of the feasible polytope is nondegenerate.

The last two assumptions will be dropped later. Without losing generality, we can describe the primal LP problem in terms of the canonical form at each iteration t of the simplex method.

LP Canonical Form

$$\begin{array}{lll} \text{PLP} & \text{minimize} & z = \bar{c}x \\ & \text{subject to} & x \in X = \{x \in R^n : \overline{A}x = \bar{b} \quad \text{and} \quad x \geq 0\} \end{array}$$

where $\overline{A} \in R^{m \times n}$, e is the vector of all one's, $\bar{b} > 0$, and

$$\overline{A} = (\, I, \quad D\,) \quad \text{and} \quad \bar{c} = (\, 0, \quad \bar{c}_D\,).$$

The initial basic feasible solution is $x^0 = (\bar{b}^T, 0)^T$ and the initial objective value is $\bar{c}x^0 = 0$. We also denote by $z^* \leq 0$ the given minimal objective value. The dual of the above linear program PLP is

DLP maximize $y\bar{b}$

subject to $y \in Y = \{y \in R^m : y\overline{A} \leq \bar{c}\}$

where the row vector $y \in R^m$. For all feasible $x \in X$ and $y \in Y$ we have by the weak duality theorem (Dantzig [3])

$$y\bar{b} \leq z^* \leq z = \bar{c}x. \tag{1}$$

The dual solution corresponding to the initial basic feasible solution is $y^0 = 0$. If $\bar{c} \geq 0$, then $y^0\overline{A} \leq \bar{c}$ is feasible and $y^0\bar{b} = \bar{c}x^0 = 0$ and the solution x^0 is optimal. Otherwise $y^0 = 0$ is an infeasible dual solution whose dual objective function shares the same value as the primal objective value $\bar{c}x^0 = 0$. We also denote by H_j the hyperplane

$$H_j = \{y \in R^m : y\bar{a}_j = \bar{c}_j\},$$

and by H_j^- the half space

$$H_j^- = \{y \in R^m : y\bar{a}_j \leq \bar{c}_j\},$$

where $\bar{a}_j$ is the jth column of $\overline{A}$. If $y^0 = 0$ is feasible for DLP, then both x^0 and y^0 solve problems PLP and DLP. Othewise, we have at least one hyperplane, say H_k, which separates 0 from Y. Obviously, the number of such separating hyperplanes are the number of $\bar{c}_j$'s with negative sign.

Let

$$\Delta = \bar{c}x - z^* = -z^*. \tag{2}$$

Then for all dual optimal feasible solutions y^* of DLP,

$$y^*\overline{A} \leq \bar{c} \quad \text{and} \quad y^*\bar{b} \geq -\Delta.$$

In particular, the first m basic columns of $\overline{A}$ state $y^* \leq 0$, so that

$$y^* \leq 0 \quad \text{and} \quad y^*\bar{b} \geq -\Delta$$

define the simplex S^t for iteration t:

$$S^t = \{y \in R^m : y \leq 0,\ y\bar{b} \geq -\Delta\}.$$

Therefore, S^t is a simplex containing all optimal dual feasible solutions y^*:

$$y^* \in S^t. \tag{3}$$

3. The Column-Eliminating Theorem

If x^* and y^* are optimal solutions to PLP and DLP respectively, then complementary slackness conditions hold:

$$(\bar{c}_j - y^*\bar{a}_j)x_j^* = 0 \quad \text{for} \quad 1 \leq j \leq n. \tag{4}$$

Hence, $\bar{c}_j - y^*\bar{a}_j > 0$ implies that $x_j^* = 0$, i.e., the jth column *is not* in any optimal basis and hence can be eliminated from the problem. A sufficient condition that $\bar{c}_j - y^*\bar{a}_j > 0$ for all dual feasible y^* is that it be true for all $y^* \in S^t$, since S^t contains all optimal dual feasible solutions as well as other y. It suffices therefore to show that the hyperplane H_j has no points in common with S^t, or that S^t is contained in the interior of H_j^-:

$$S^t \subset Int(H_j^-). \tag{5}$$

However, S^t only has $m + 1$ vertices, namely

$$V^0 = 0, \; V^i = (-\Delta/\bar{b}_i)U_i \quad \text{for} \quad 1 \leq i \leq m, \tag{6}$$

where U_i is the ith unit vector. Therefore, $S^t \subset Int(H_j^-)$ *if and only if* these vertices V^i are strictly in the interior of the half space H_j^-. Thus, we have

Theorem 1

If

$$\bar{c}_j - y\bar{a}_j > 0 \quad \text{for} \quad y = V^i, \; i = (0, 1, ..., m), \tag{7}$$

then the jth column of $\overline{A}$ *is not* in any optimal basis for PLP.

Proof. If (7) is true, then $\bar{c}_j - y\bar{a}_j > 0$ for all $y = \sum \lambda_i V^i = S^t$ where $\lambda_i \geq 0, \; \sum \lambda_i = 1$. Since all optimal dual feasible solutions $y^* \in S^t$,

$$\bar{c}_j - y^*\bar{a}_j > 0 \Longrightarrow x_j^* = 0.$$

Hence, the proof simply follows from the complementary slackness conditions discussed above. Q.E.D.

Furthermore, we can explicitly calculate the minimum of those $m+1$ values in Theorem 1. Letting $\bar{a}_{ij}$ be the ith component of column $\bar{a}_j$, we derive

Corollary 1 (nondegenerate case)

If

$$\bar{c}_j + \Delta \min(0, \min_{1 \le i \le m} (\frac{\bar{a}_{ij}}{\bar{b}_i})) > 0, \tag{8}$$

then the jth column of $\overline{A}$ *is not* in any optimal basis for PLP.

In order to apply Theorem 1, the values $\bar{a}_{ij}$, $\bar{b}_i$, and $\bar{c}_j$ must be available from the canonical form. In the revised simplex method only $\bar{b}_i$ and the inverse of the basis in either explicit or factorized form are available at the start of iteration t. It is necessary to compute $\bar{c}_j$ but it is too costly to compute $\bar{a}_{ij}$ for every column j except for the selected incoming column $j = s$. However, at the end of the iteration t, the updated values $\bar{a}_{ij_r}$ for the outgoing column j_r of iteration t are readily at hand for iteration $t+1$ and Theorem 1 may be applied to j_r. For example, if (8) holds for updated $\bar{a}_{ij_r}$, $\bar{b}_i$, and $\bar{c}_{j_r}$, then column j_r will never reenter the basis. This results in the following corollary.

Corollary 2 (nondegenerate case)

Let the sth column be the incoming column in the simplex method and $\bar{a}_{rs}$ be the pivot. In either of the following two cases,

1) all $\bar{a}_{is}$, except $\bar{a}_{rs}$, are nonpositive;

2)

$$\frac{\bar{b}_i}{\bar{a}_{is}} > \frac{\Delta}{|\bar{c}_s|} \quad \text{for all} \quad \bar{a}_{is} > 0, \quad i \neq r,$$

the outgoing column j_r can be eliminated.

Proof. Note that

$$\text{updated } \Delta = \Delta + \frac{\bar{b}_r \bar{c}_s}{\bar{a}_{rs}} \ge 0;$$

$$\text{updated } \bar{c}_{j_r} = \frac{-\bar{c}_s}{\bar{a}_{rs}} > 0;$$

$$\text{updated } \bar{b}_r = \frac{\bar{b}_r}{\bar{a}_{rs}} > 0;$$

$$\text{updated } \bar{b}_i = \bar{b}_i - \frac{\bar{b}_r \bar{a}_{is}}{\bar{a}_{rs}} > 0, \ i \neq r;$$

$$\text{updated } \bar{a}_{rj_r} = \frac{1}{\bar{a}_{rs}} > 0;$$

$$\text{updated } \bar{a}_{ij_r} = \frac{-\bar{a}_{is}}{\bar{a}_{rs}}, \ i \neq r.$$

Substituting the above updated values into (8), the condition for eliminating column j_r becomes

$$\frac{-\bar{c}_s}{\bar{a}_{rs}} + (\Delta + \frac{\bar{b}_r \bar{c}_s}{\bar{a}_{rs}}) \min(0, \min_{i \neq r}(\frac{-\bar{a}_{is}/\bar{a}_{rs}}{\bar{b}_i - \bar{b}_r \bar{a}_{is}/\bar{a}_{rs}}), \frac{1/\bar{a}_{rs}}{\bar{b}_r/\bar{a}_{rs}}) > 0. \tag{9}$$

Obviously, if $\bar{a}_{is} \leq 0$ for all $i \neq r$, the second term is zero and the first term is positive. Hence column j_r can be dropped in case 1.

In case 2 of Corollary 2,

$$\min(0, \min_{i \neq r}(\frac{-\bar{a}_{is}/\bar{a}_{rs}}{\bar{b}_i - \bar{b}_r \bar{a}_{is}/\bar{a}_{rs}}), \frac{1/\bar{a}_{rs}}{\bar{b}_r/\bar{a}_{rs}}) = (\frac{-1}{\bar{a}_{rs}}) \frac{1}{\min_{i \neq r, \bar{a}_{is} > 0}(\frac{\bar{b}_i}{\bar{a}_{is}} - \frac{\bar{b}_r}{\bar{a}_{rs}})}.$$

Thus, multiplying (9) by $\bar{a}_{rs}$,

$$-\bar{c}_s - \frac{\Delta + \bar{b}_r \bar{c}_s/\bar{a}_{rs}}{\min_{i \neq r, \bar{a}_{is} > 0}(\frac{\bar{b}_i}{\bar{a}_{is}} - \frac{\bar{b}_r}{\bar{a}_{rs}})} > 0,$$

or

$$\min_{i \neq r, \bar{a}_{is} > 0}(\frac{\bar{b}_i}{\bar{a}_{is}} - \frac{\bar{b}_r}{\bar{a}_{rs}}) > \frac{\Delta + \bar{b}_r \bar{c}_s/\bar{a}_{rs}}{-\bar{c}_s},$$

which is equivalent to

$$\frac{\bar{b}_i}{\bar{a}_{is}} > \frac{\Delta}{|\bar{c}_s|} \quad \text{for all} \quad \bar{a}_{is} > 0 \quad i \neq r. \qquad \textit{Q.E.D.}$$

One should note that if two rows tie for pivot, the outgoing column can not be eliminated since degeneracy occurs at iteration $t+1$ (Dantzig [4]). We now remove the nondegeneracy assumption, i.e., we allow that $\bar{b} \geq 0$ instead of $\bar{b} > 0$. In this case, S^t may not be bounded. However, Theorem 1 is still valid if we define

$$V^i = \begin{cases} (-\Delta/\bar{b}_i)U_i & \text{if } \bar{b}_i > 0 \\ (-\infty)U_i & \text{if } \bar{b}_i = 0, \end{cases}$$

and Corollary 1 can be modified as

Corollary 3 (degenerate or nondegenerate case)

If for some i, $\bar{b}_i = 0$ and $\bar{a}_{ij} < 0$, then don't eliminate column j; otherwise if

$$\bar{c}_j + \Delta \min(0, \min_{1 \le i \le m, \bar{b}_i > 0} (\frac{\bar{a}_{ij}}{\bar{b}_i})) > 0,$$

then the jth column of $\overline{A}$ *is not* in any optimal basis for PLP.

Similarly, Corollary 2 can be modified as

Corollary 4 (degenerate or nondegenerate case)

Let sth column be the incoming column in the simplex method and $\bar{a}_{rs}$ be the pivot. In either of the following two cases,

1) all $\bar{a}_{is}$, except $\bar{a}_{rs}$, are nonpositive;

2)

$$\frac{\bar{b}_i}{\bar{a}_{is}} > \max(\frac{\Delta}{|\bar{c}_s|}, \frac{\bar{b}_r}{\bar{a}_{rs}}) \quad \text{for all} \quad \bar{a}_{is} > 0, \quad i \neq r,$$

the outgoing column j_r can be eliminated.

4. The Lower Bound Generating Technique

Until now we have assumed that the minimal objective value z^* is known in advance. Actually, all of the above theorems and corollaries are still valid if z^* of (2) is replaced by a lower bound z^0 for z^*. At each step of the simplex method, we can update z^0 to a possibly higher lower bound by using the following techniques.

Theorem 2

Let $\overline{a}_i$ be the ith row of $\overline{A}$,

$$\delta_i = \begin{cases} \overline{b}_i \max_{\delta \in R}\{\delta : \overline{c} - \delta \overline{a}_i \geq 0\}, & \text{if } \{\delta : \overline{c} - \delta \overline{a}_i \geq 0\} \neq \emptyset; \\ -\infty, & \text{otherwise} \end{cases} \tag{10}$$

and

$$\Delta = - \max_{1 \leq i \leq k} \delta_i.$$

Then for all feasible $x \in X$,

$$z = \overline{c}x \geq -\Delta.$$

Proof. If Δ is finite, say $\Delta = -\delta_k$, then

$$\overline{c} - \delta_k \overline{a}_k = \overline{c} - \delta_k U_k \overline{A} \geq 0,$$

i.e., $\delta_k U_k$ is a feasible solution for DLP. Therefore, from (1),

$$-\Delta = \delta_k U_k \overline{b} \leq \overline{c}x = z$$

for all feasible $x \in X$. Q.E.D.

The following corollary resembles Theorem 2 by looking for a dual feasible solution that has the form δe^T.

Corollary 5

Let

$$\Delta = \begin{cases} -(e^T \overline{b}) \max_{\delta \in R}\{\delta : \overline{c} - \delta e^T \overline{A} \geq 0\}, & \text{if } \{\delta : \overline{c} - \delta e^T \overline{A} \geq 0\} \neq \emptyset; \\ \infty, & \text{otherwise .} \end{cases}$$

Then for all feasible $x \in X$,

$$z = \overline{c}x \geq -\Delta.$$

Thus, Δ can be calculated using a ratio test on $\overline{A}$ against $\overline{c}$.

5. A Numerical Example

Suppose that the simplex tableau is given as follows (Dantzig [3] pp. 97):

Rows	Col 1	Col 2	Col 3	Col 4	Col 5	RHS
Row 1	-3/2	7/8	0	-3/8	1	1/2
Row 2	1/2	-3/8	1	-1/8	0	3/2
Cost	12	-1	0	2	0	0

Using (10), we obtain $\delta_1 = -4/7$ and $\delta_2 = -\infty$. Hence, $\Delta = 4/7$ from Theorem 2.

Now using Corollary 1, we have

$$\bar{c}_1 + \Delta \min(0, \min_{1 \le i \le 2} (\frac{\bar{a}_{i1}}{\bar{b}_i})) = 12 + (4/7)(-3) > 0$$

and

$$\bar{c}_4 + \Delta \min(0, \min_{1 \le i \le 2} (\frac{\bar{a}_{i4}}{\bar{b}_i})) = 2 + (4/7)(-3/4) > 0.$$

Therefore, Columns 1 and 4 can be eliminated from the computational tableau.

Moreover, only the pivot $\bar{a}_{12} = 7/8$ in the incoming column (Col 2) is positive. Using Corollary 4, the outgoing column (Col 5) can be eliminated from the tableau as well. Thus, we have only two columns (Col 2 and Col 3) left, which form the optimal basis.

6. Conclusion

We have proposed column-eliminating and lower bound updating techniques in the simplex method for linear programming. A pricing criterion is developed on checking the intersection between dual hyperplanes and the dual simplex containing all of the optimal dual feasible solutions. Under this criterion, some columns may be identified early as the optimal nonbasic columns; therefore they can be eliminated in the course of the simplex method. As the simplex method iterates, the working constraint matrix ultimately reduces to only those columns that are in some optimal basis.

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